Introduction
000

Main Theorem
000000

Numerical semigroup rings
0000

Reference
00

On higher dimensional Teter rings
via the canonical trace ideal

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Introduction
000

Main Theorem
000000

Numerical semigroup rings
0000

Reference
00

Table of Contents

- Introduction

- Main Theorem

- Numerical semigroup rings

- Reference

Introduction
●○○

Main Theorem
○○○○○○

Numerical semigroup rings
○○○○

Reference
○○

Teter rings

In this talk, we assume a ring R is not Gorenstein.

Definition (W.Teter, 1974 [1])

Let $(R, \mathfrak{m})$ be an Artinian local ring with canonical module ω_R.
R is called a *Teter ring* if there exists $\varphi \in \mathrm{Hom}_R(\omega_R, R)$ such that $\varphi(\omega_R) = \mathfrak{m}$.

Teter rings were formulated in the higher-dimensional case.

Definition 1 (T. J. Puthenpurakal, 2025 [2])

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay **local ring**.
Then, *R* is called a *Teter ring* if there exist a Gorenstein local ring *S* and a
surjective ring homomorphism $\varphi : S \to R$ such that $\mathrm{e}(R) - \mathrm{e}(S) = 1$ where
$\mathrm{e}(R)$ denotes multiplicity of *R*.

Remark

If *R* is an Artinian local ring, then we can check Puthenpurakal's definition is
equivalent to Teter's definition.

Introduction
○●○

Main Theorem
○○○○○○

Numerical semigroup rings
○○○○

Reference
○○

Setting

Throughout:

(1) Let $R = \bigoplus_{i \geq 0} R_i$ be a Cohen–Macaulay graded ring with a field R_0 and the maximal graded ideal $\mathfrak{m}$;

(2) We say that R is (resp. *semi-*) standard graded if R is (resp. a finitely generated as an $R_0[R_1]$-module) $R_0[R_1]$;

(3) ω_R : the graded canonical module ;
$\mathrm{tr}(\omega_R) := \sum_{\varphi \in \mathrm{Hom}_R(\omega_R, R)} \mathrm{Im}(\varphi) \subset R$: the canonical trace ideal ;
$r(R)$: the *Cohen–Macaulay type* ;
a_R : the *a-invariant* ;
$\mathrm{indeg}(\mathfrak{m}) := \min\{i \in \mathbb{Z} : [\mathfrak{m}]_i \neq 0\}$;
$\mathrm{codim}(R) := \mathrm{embdim}(R) - \dim(R)$;

(4) $r_0(R) := \dim_{R_0}([\omega_R]_{-a_R})$;

(5) We say that R is *level* if $r(R) = r_0(R)$;

(6) We assume R is **NOT** Gorenstein.

Teterness for graded rings

Definition 2 (Miyashita–O., 2025)

We say that R is a *Teter ring* if there exists a graded R-homomorphism $\varphi : \omega_R \to R$ such that it satisfies either one of the following conditions:

(1) $\varphi(\omega_R) = \mathfrak{m}$ (this happens only when $\dim R = 0$);

(2) φ is injective and $\text{embdim}\,(R/\varphi(\omega_R)) \leq \dim R$.

Remark

If R is a domain, the following conditions are equivalent:

(1) R is a *Teter ring* in the sense of Definition 2;

(2) $R_{\mathfrak{m}}$ is a *Teter ring* in the sense of Definition 1.

Introduction
ooo

Main Theorem
●ooooo

Numerical semigroup rings
oooo

Reference
oo

Main Theorem

Theorem (Miyashita–O., 2025)

Assume that $\dim(R) > 0$. Consider the following conditions:

(1) R is level, $r(R) \geq \operatorname{codim}(R)$, and $[\operatorname{tr}(\omega_R)]_{\operatorname{indeg}(\mathfrak{m})}$ contains a n.z.d.;

(2) $[\omega_R]_{-a_R}$ contains a torsion-free element (e.g., R is a domain), $r_0(R) \geq \operatorname{codim}(R)$, and $[\operatorname{tr}(\omega_R)]_{\operatorname{indeg}(\mathfrak{m})}$ contains a n.z.d.;

(3) R is a level Teter ring;

(4) R is a Teter ring (which implies $r(R) = \operatorname{codim}(R)$).

Then the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold.
Moreover, if R is a standard graded, all conditions are equivalent.

Remark

We say an element $x \in \omega_R$ is *torsion-free* if $rx = 0$ implies $r = 0$ for all $r \in R$.

Introduction
000

Main Theorem
0●0000

Numerical semigroup rings
0000

Reference
00

As a zero-dim analogue of our main theorem, one may ask the following.
Assume R is standard graded and $\dim(R) = 0$. Consider the following :

(1) R is level, $r(R) \geq \mathrm{codim}(R)$, and $\mathrm{tr}(\omega_R) = \mathfrak{m}$;

(2) R is a Teter ring.

Then we can check that $(2) \Rightarrow (1)$. Does $(1) \Rightarrow (2)$ also hold ?
Unfortunately, there is a counterexample to the above Question.

Example

$R = \mathbb{Q}[x_1, x_2, x_3, x_4]/(x_1^3, x_2^2, x_3^3, x_4^2, x_1 x_2, x_1 x_3^2, x_1^2 x_3, x_2 x_4, x_3 x_4, x_2 x_3 - x_1 x_4)$.
We can check that R is level, that $r(R) = \mathrm{codim}(R) \ (= 4)$, and that
$\mathrm{tr}(\omega_R) = \mathfrak{m}$, whereas R is not Teter.

Introduction
000

Main Theorem
000●000

Numerical semigroup rings
0000

Reference
00

Corollary (Miyashita–O., 2025)

Assume that R is a standard graded nearly Gorenstein ring.
Then R is a *Teter ring*, if R satisfies one of the following conditions:

(1) R is level with $r(R) = \operatorname{codim}(R)$ and $\dim(R) > 0$;

(2) R is a domain with $\operatorname{codim}(R) = 2$ and $\dim(R) \geq 2$;

(3) R has minimal multiplicity;

(4) R is a Stanley–Reisner ring;

(5) R is a domain whose *h*-vector is $h(R) = (1, a, a)$ for some $a > 0$.

Introduction
○○○

Main Theorem
○○○○●○○

Numerical semigroup rings
○○○○

Reference
○○

Notice that every standard graded Teter ring R is level. In particular, we have

$$r(R) - r_0(R) = 0.$$

Example

For any $a \in \mathbb{Z}_{>0}$, there exists a (non-standard) semi-standard graded Teter ring R with

$$r(R) - r_0(R) = a.$$

Indeed, we consider the affine semigroup ring $R := \mathbb{Q}[S]$, where

$$S = \langle \{(2i, 2a+2-2i) : 0 \le i \le a+1\} \cup \{(2j+1, 4a-2j+3) : 0 \le j \le a\} \rangle.$$

Then we can check R is a Teter ring by the definition.

Upper bound for Cohen–Macaulay type

Proposition (Miyashita–O., 2025)

Assume that $[\operatorname{tr}(\omega_R)]_{\operatorname{indeg}(\mathfrak{m})}$ contains a n.z.d. of R.
Then the following conditions hold:

(1) If R is level, then $r(R) \leq \operatorname{codim}(R)$.

(2) If $r_0(R) \geq \operatorname{codim}(R)$, then it is level and $r(R) = \operatorname{codim}(R)$.

Corollary (Miyashita–O., 2025)

Assume R is standard graded nearly Gorenstein with $\dim(R) \geq 2$.
If R satisfies $\operatorname{codim}(R) = 3$ and $r_0(R) \neq 2$, then R is level and $r(R) = 3$.

0-dimensional counterexample

$A = \mathbb{Q}[x, y, z]/(xz^2 - y^3,\ x^3 + xy^2 - y^2z,\ x^2y + y^3 - z^3, x^2 + z^2)$ is a non-level standard graded nearly Gorenstein ring with $\operatorname{codim}(R) = 3$ and $r(R) = 2$.

Introduction
000

Main Theorem
000000●

Numerical semigroup rings
0000

Reference
00

Some operations and Teter rings

Proposition (Miyashita–O., 2025)

Assume that A and B are Cohen–Macaulay generically Gorenstein standard graded rings with $\dim(A) = \dim(B) = 1$ at least one of which is not regular. The following are equivalent :

(1) $A \times_{R_0} B$ is a Teter ring ;

(2) Both A and B have minimal multiplicity.

Proposition (Miyashita–O., 2025)

Assume R is a non-Gorenstein generically Gorenstein standard graded ring with minimal multiplicity. If either $\dim(R) = 1$ with $e(R) \geq 3$ or $\mathrm{tr}(\omega_R) = \mathfrak{m}$ with $\dim(R) = 2$, then $R^{(k)}$ is Teter for every $k \geq 2$.

Introduction
000

Main Theorem
000000

Numerical semigroup rings
●000

Reference
00

Setting

We consider the Teterness of numerical semigroup rings as the typical class of non-standard graded rings.

In this section:

(1) H : a numerical semigroup.

(2) We can find the minimal generators $H = \langle a_1, \ldots, a_n \rangle$ where $\gcd(a_1, \ldots, a_n) = 1$ and $0 < a_1 < \cdots < a_n$.

(3) $k[H] := k[t^{a_1}, \ldots, t^{a_n}]$: the numerical semigroup ring.

(4) $\mathrm{PF}(H) = \{\alpha \in \mathbb{Z} \mid \forall h \in H, \alpha + h \in H\}$: Pseudo-Frobenius numbers. This is related to the generators of the canonical module;

$$\omega_{R_H} = \sum_{\alpha \in \mathrm{PF}(H)} t^{-\alpha} R_H$$

Characterization

Proposition (Miyashita–O., 2025)

The following are equivalent.

(1) $k[H]$ is a *Teter ring* ;

(2) $\mathrm{PF}(H)$ consists of $(n-1)$ elements $\alpha_1 < \cdots < \alpha_{n-1}$, and $\exists N \in \{1, \ldots, n\}$ s.t. $2 \leq \forall j \leq n$

$$\alpha_{n-1} + a_1 + \delta = \begin{cases} a_j + \alpha_{n-j} & (j < N) \\ a_j + \alpha_{n+1-j} & (j > N) \end{cases}$$

where

$$\delta = \begin{cases} a_2 - a_1 & (N = 1) \\ 0 & (N \neq 1) \end{cases}$$

Remark

If H has minimal multiplicity, then it is only necessary to verify $N = 1$ or n.

Introduction
000

Main Theorem
000000

Numerical semigroup rings
0000

Reference
00

Corollary (Miyashita–O., 2025)

We assume that H has minimal multiplicity. Then,

$$[\mathrm{tr}(\omega_{R_H})]_{a_1} \neq (0) \Leftrightarrow k[H] \text{ is almost Gorenstein} \Leftrightarrow k[H] \text{ is Teter and } \delta = 0.$$

hold true.

Corollary (Miyashita–O., 2025)

Let $H = \langle n_1, n_2, n_3 \rangle$ be a non-symmetric numerical semigroup with embedding dimension 3. Then, the following holds true.

(1) If $k[H]$ is nearly Gorenstein but not almost Gorenstein, then $k[H]$ is *Teter*.

(2) When $k[H]$ is almost Gorenstein, Hilbert-Burch matrix of $k[H]$ can be expressed as $A = \begin{pmatrix} x_1^a & x_2^b & x_3^c \\ x_2 & x_3 & x_1 \end{pmatrix}$ $(a, b, c > 0)$.

Then, $k[H]$ is *Teter* if and only if $\min\{a, b, c\} = 1$.

Introduction
○○○

Main Theorem
○○○○○○

Numerical semigroup rings
○○○●

Reference
○○

Examples

(1) If $\mathrm{emb}(H) = 4$, the previous corollary does not hold true.

#PF(H) = 3 case.		Teter
	$\langle 7, 10, 12, 13 \rangle$	$\langle 7, 10, 12, 13 \rangle$
minimal multiplicity		$\langle 7, 11, 15, 19 \rangle$
$\langle 4, 7, 9, 10 \rangle$	$\langle 4, 9, 10, 11 \rangle$	$\langle 4, 5, 6, 7 \rangle$
	almost Gorenstein	$\langle 8, 9, 11, 15 \rangle$
	nearly Gorenstein	$\langle 11, 12, 14, 15 \rangle$

(2) (Generalized arithmetic sequence)
For $n > 1$, $a, s, d > 0$ s.t. $\gcd(a, d) = 1$, $H := \langle a, sa + d, \ldots, sa + nd \rangle$.
If the remainder of a divided by n is 1, then $k[H]$ is a Teter ring that is not necessarily almost Gorenstein.

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Introduction
ooo

Main Theorem
oooooo

Numerical semigroup rings
oooo

Reference
o●

Thank you for your attention.